Dynamics of the Universal Confining String Theory on the Loop Space

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Abstract

Starting with the representation of the Wilson average in the Euclidean 4D compact QED as a partition function of the Universal Confining String Theory, we derive for it the corresponding loop equation, alternative to the familiar one.

In the functional momentum representation the obtained equation decouples into two independent ones, which describe the dynamics of the transverse and longitudinal components of the area derivative of the Wilson loop. At some critical value of the momentum discontinuity, which can be determined from a certain equation, the transverse component does not propagate.

Next, we derive the equation for the momentum Wilson loop, where on the left-hand side stands the sum of the squares of the momentum discontinuities, multiplied by the loop, which describes its free propagation, while the right-hand side describes the interaction of the loop with the functional vorticity tensor current.

Finally, using the method of inversion of the functional Laplacian, we obtain for the Wilson loop in the coordinate representation a simple Volterra type-II linear integral equation, which can be treated perturbatively.

1. Introduction

Recently a new progress in the string representation of the confining phase of gauge theories was achieved by the development of the so-called Universal Confining String Theory (UCST)[1], whose partition function is nothing else than the Wilson loop in the 3D compact QED. It was argued in Ref.1 that the summation over the branches of the multivalued action of the UCST corresponds to the summation over the string world sheets, which is a new step in the understanding of the connection between gauge fields and strings.

In Ref.2 the action of the UCST in 4D was derived by performing the exact duality transformation, and the low-energy limit of this theory with the θ-term included was investigated. This low-energy expansion was already discussed in Ref.1 (in the low-energy limit the Wilson loop may be evaluated exactly), and in Ref.2 this program was elaborated out, so that the string tension of the Nambu-Goto term and the coupling constant of the rigidity term were calculated explicitly. The sign of the obtained rigid string coupling constant is negative, which means that this low-energy string effective action is stable (see for example Ref.3). Also the string θ-term appeared in the low-energy expansion due to the field-theoretical one.

The most important property of the UCST lies in the observation, proposed in Ref.1, that this string theory describes all the gauge theories in such a way that the contribution to the partition function of the UCST, going from the group $U(N)$, enters with the 't Hooft factor $N^{-\chi}$, where χ is the Euler character of the string world sheet. This conjecture seems to bee justified by proving the fact that the partition function of the UCST satisfies loop equations[4,5] modulo contact terms, which can't be analized since their structure depends upon the topology of the string world sheet.

In this letter we shall use the representation of the Wilson average in the Euclidean 4D compact QED as a partition function of the UCST in order to derive for it the corresponding loop equation, alternative to the familiar one. Thus, the equation, which will be obtained, is nothing else than the equation of motion of the partition function of the UCST, written in the loop space. The low-energy limit of the UCST, i.e. the case when $\frac{H^2_{\mu\nu\lambda}}{m^6 e^6 \exp\left(\frac{const}{e^2}\right)} \ll 1$, where $H_{\mu\nu\lambda}$ is the strength tensor of the Kalb-Ramond field, m is the mass of this field, and e is the coupling constant, corresponds to an obvious simplification of this loop equation.

Next, we shall perform the functional Fourier transformation[6,7] and rewrite this equation in the momentum representation, where the functional differential operator, acting onto the area derivative of the Wilson loop, may be easily inverted, and the resulting equation decouples into a system of two independent ones, which describe the dynamics of the transverse and longitudinal components of the area derivative of the loop. We shall see that at some critical value of the discontinuity of the momentum loop, which can be determined from the corresponding equation, the transverse part of the loop area derivative does not propagate. In the low-energy limit of the UCST this equation possesses only a spurious solution, and therefore the transverse component always propagates in this limit. After that, we shall bring the momentum loop equation to such a form, that on its L.H.S. will stand the sum of the squares of the momentum discontinuities, multiplied by the momentum loop, which is known to describe the free propagation of the loop[6], while the R.H.S. of this equation will describe the interaction of the loop with the functional vorticity tensor current.

Finally, we shall return to the coordinate representation and make use of the method of inversion of the functional Laplacian[8], so that the resulting equation will be simply a Volterra type-II linear integral equation, which can be investigated perturbatively.

All the points, mentioned above, will be the topic of the next Section.

The main results of the letter are summarized in the Conclusion.

2. Loop equation for the partition function of the UCST and its investigation

The partition function of the UCST, which is nothing else than the Wilson average in the Euclidean 4D compact QED, has the form[1,2]

$$W[\mathbf{x}] = \langle \Phi(B_{\alpha\beta}) \rangle$$

(we use the notations, emphasizing that the Wilson loop is a functional, defined on the loop space; here $\mathbf{x} \equiv x_\mu(s), 0 \le s \le 1$, is an element of this space), where $\Phi(B_{\alpha\beta}) = \exp(i \int d\sigma_{\alpha\beta} B_{\alpha\beta})$, $< ... >\equiv \int DB_{\mu\rho} e^{-S}(...)$, and the action of the Kalb-Ramond field $B_{\mu\nu}$ reads as follows

$$S = \int d^4x \left[\frac{2}{3} \Lambda_0 H_\mu arcsh \left(\frac{2}{3z\Lambda_0^3} H_\mu \right) - z\Lambda_0^4 \sqrt{1 + \frac{4}{9z^2\Lambda_0^6} H_\mu^2} + \frac{1}{4e^2} B_{\mu\nu}^2 \right]. \tag{1}$$

Here $H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}$ is a strength tensor of the field $B_{\mu\nu}$, $H_\mu = \varepsilon_{\mu\nu\alpha\beta} H_{\nu\alpha\beta}$, Λ_0 is a cutoff, which is necessary in 4D, e is a dimensionless coupling constant, and $z \sim \exp\left(-\frac{const}{e^2} \right)$.

The equation of motion $\int DB_{\mu\rho} \frac{\delta}{\delta B_{\nu\lambda}(x)} e^{-S} \Phi(B_{\alpha\beta}) = 0$ has the form

$$\left\langle \left[\frac{1}{2\Lambda^2} \left(\partial^2 B_{\nu\lambda} + \partial_\mu \partial_\nu B_{\lambda\mu} + \partial_\mu \partial_\lambda B_{\mu\nu} \right) + \sqrt{1 - \frac{z}{1536\Lambda^6} H_{\rho\sigma\zeta}^2} \left(\frac{1}{2e^2} B_{\nu\lambda} - iT_{\nu\lambda} \right) \right] \Phi(B_{\alpha\beta}) \right\rangle = 0, \tag{2}$$

where $T_{\nu\lambda}(x) = \int d^2\xi \varepsilon^{ab} (\partial_a x_\nu(\xi)) (\partial_b x_\lambda(\xi)) \delta(x - x(\xi))$ is the vorticity tensor current, and $\Lambda \equiv \frac{\Lambda_0}{4} \sqrt{z}$. By virtue of Eq.(2) one gets the following loop equation

$$\left[\frac{1}{2\Lambda^2} \left(\partial^{x(\sigma)\,2} \frac{\delta}{\delta\sigma_{\nu\lambda}(x(\sigma))} + \partial_\mu^x \partial_\nu^x \frac{\delta}{\delta\sigma_{\lambda\mu}(x)} + \partial_\mu^x \partial_\lambda^x \frac{\delta}{\delta\sigma_{\mu\nu}(x)} \right) + \right.$$

$$\left. + \sqrt{1 + \frac{z}{512\Lambda^6} \left[\left(\partial_\alpha^x \frac{\delta}{\delta\sigma_{\beta\gamma}(x)} \right)^2 + 2 \left(\partial_\alpha^x \frac{\delta}{\delta\sigma_{\beta\gamma}(x)} \right) \left(\partial_\beta^x \frac{\delta}{\delta\sigma_{\gamma\alpha}(x)} \right) \right]} \left(\frac{1}{2e^2} \frac{\delta}{\delta\sigma_{\nu\lambda}(x)} + T_{\nu\lambda}[\mathbf{x}] \right) \right] W[\mathbf{x}] = 0, \tag{3}$$

where $\partial_\mu^{x(\sigma)} \equiv \int\limits_{\sigma-0}^{\sigma+0} d\sigma' \frac{\delta}{\delta x_\mu(\sigma')}$, and $T_{\nu\lambda}$ becomes a functional defined on the loop space.

Notice, that in the low-energy limit of the UCST action (1) takes the form $S = \int d^4x \left(-\frac{1}{12\Lambda^2} H_{\mu\nu\lambda}^2 + \frac{1}{4e^2} B_{\mu\nu}^2 \right)$, which yields the mass of the Kalb-Ramond field in this limit $m = \frac{\Lambda}{e}$, and therefore the validity of the low-energy approximation means that $\frac{H_{\mu\nu\lambda}^2}{z^2 e^6 m^6} \ll 1$. If this inequality holds true, the square root, standing on the L.H.S. of Eq.(3), replaces by unity.

Let us now rewrite Eq.(3) in the momentum representation by making use of the functional Fourier transformation[6,7], which for an arbitrary functional $A[\mathbf{x}]$ is defined as $A[\mathbf{p}] = \int D\mathbf{x} \exp\left(i \int\limits_0^1 d\sigma p_\mu \dot{x}_\mu \right) A[\mathbf{x}]$, where $D\mathbf{x} \equiv \mathcal{D}\mathbf{x}\delta(x(0) - x(1))\delta(x(0) - const)$, and $\mathcal{D}\mathbf{x}$ stands for the ordinary functional measure. Taking into account that[4] $\frac{\delta}{\delta\sigma_{\mu\nu}(x(\sigma))} = \int\limits_{-0}^{+0} d\tau \tau \frac{\delta^2}{\delta x_\mu\left(\sigma+\frac{1}{2}\tau\right)\delta x_\nu\left(\sigma-\frac{1}{2}\tau\right)}$, we arrive at the following equation for the momentum Wilson loop

$$\left(\frac{e^2}{Q[\mathbf{p}]}\frac{(\Delta p)^2}{\Lambda^2}\mathbf{P}_{\mu\rho,\lambda\nu} - \mathbf{1}_{\mu\rho,\lambda\nu}\right)\int\limits_{-0}^{+0}d\tau\tau\dot{p}_\nu\left(\sigma + \frac{1}{2}\tau\right)\dot{p}_\lambda\left(\sigma - \frac{1}{2}\tau\right)W[\mathbf{p}] = 2e^2\int D\mathbf{p}'T_{\mu\rho}[\mathbf{p} - \mathbf{p}']W[\mathbf{p}'].$$

$$\tag{4}$$

Here $\Delta p_\mu \equiv \Delta p_\mu(\sigma) = p_\mu(\sigma + 0) - p_\mu(\sigma - 0)$ is the momentum discontinuity,

$$Q[\mathbf{p}] \equiv \left(1 + \frac{z}{512\Lambda^6}\Delta p_\alpha\int\limits_{-0}^{+0}d\tau\int\limits_{-0}^{+0}d\tau'\tau\tau'\dot{p}_\beta\left(\sigma + \frac{1}{2}\tau\right)\dot{p}_\gamma\left(\sigma - \frac{1}{2}\tau\right)\dot{p}_\gamma\left(\sigma - \frac{1}{2}\tau'\right)\left(2\Delta p_\beta\cdot\right.$$

$$\left.\cdot\dot{p}_\alpha\left(\sigma + \frac{1}{2}\tau'\right) - \Delta p_\alpha\dot{p}_\beta\left(\sigma + \frac{1}{2}\tau'\right)\right)\right)^{\frac{1}{2}}, \quad \mathbf{P}_{\mu\rho,\lambda\nu} = \frac{1}{2}\left(\mathcal{P}_{\mu\lambda}\mathcal{P}_{\rho\nu} - \mathcal{P}_{\mu\nu}\mathcal{P}_{\rho\lambda}\right), \quad \mathcal{P}_{\mu\nu} \equiv \delta_{\mu\nu} - \frac{\Delta p_\mu\Delta p_\nu}{(\Delta p)^2},$$

$$\mathbf{1}_{\mu\rho,\lambda\nu} \equiv \frac{1}{2}\left(\delta_{\mu\lambda}\delta_{\rho\nu} - \delta_{\mu\nu}\delta_{\rho\lambda}\right),$$

so that $\mathbf{P}_{\mu\rho,\lambda\nu}$ and $(\mathbf{1} - \mathbf{P})_{\mu\rho,\lambda\nu}$ are the projectors onto the transverse and longitudinal degrees of freedom of the momentum area derivative of the Wilson loop respectively, which satisfy the following relations $\mathbf{P}^2 = \mathbf{P}$, $(\mathbf{1} - \mathbf{P})^2 = \mathbf{1} - \mathbf{P}$, $\mathbf{P}(\mathbf{1} - \mathbf{P}) = 0$. The operator, standing in front of the integral on the L.H.S. of Eq.(4), can be easily inverted, which yields

$$\int\limits_{-0}^{+0}d\tau\tau\dot{p}_\nu\left(\sigma + \frac{1}{2}\tau\right)\dot{p}_\lambda\left(\sigma - \frac{1}{2}\tau\right)W[\mathbf{p}] =$$

$$= 2e^2\left(\frac{(\Delta p)^2}{(\Delta p)^2 - \frac{Q[\mathbf{p}]}{e^2}\Lambda^2}\mathbf{P}_{\lambda\nu,\mu\rho} - \mathbf{1}_{\lambda\nu,\mu\rho}\right)\int D\mathbf{p}'T_{\mu\rho}[\mathbf{p} - \mathbf{p}']W[\mathbf{p}'].\tag{5}$$

Eq.(5) splits into a pair of two independent equations for the transverse and longitudinal components of the momentum area derivative of the Wilson loop, which read correspondingly

$$\mathbf{P}_{\mu\rho,\lambda\nu}\int\limits_{-0}^{+0}d\tau\tau\dot{p}_\nu\left(\sigma + \frac{1}{2}\tau\right)\dot{p}_\lambda\left(\sigma - \frac{1}{2}\tau\right)W[\mathbf{p}] =$$

$$= 2Q[\mathbf{p}]\frac{\Lambda^2}{(\Delta p)^2 - \frac{Q[\mathbf{p}]}{e^2}\Lambda^2}\mathbf{P}_{\mu\rho,\lambda\nu}\int D\mathbf{p}'T_{\lambda\nu}[\mathbf{p} - \mathbf{p}']W[\mathbf{p}'],\tag{6}$$

$$(\mathbf{1} - \mathbf{P})_{\mu\rho,\lambda\nu}\int\limits_{-0}^{+0}d\tau\tau\dot{p}_\nu\left(\sigma + \frac{1}{2}\tau\right)\dot{p}_\lambda\left(\sigma - \frac{1}{2}\tau\right)W[\mathbf{p}] = -2e^2(\mathbf{1} - \mathbf{P})_{\mu\rho,\lambda\nu}\int D\mathbf{p}'T_{\lambda\nu}[\mathbf{p} - \mathbf{p}']W[\mathbf{p}'].$$

$$\tag{7}$$

We see from Eq.(6) that when the momentum discontinuity satisfies the equation

$$(\Delta p)^2 = \frac{Q[\mathbf{p}]}{e^2}\Lambda^2,\tag{8}$$

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the transverse component does not propagate. In the low-energy limit of the UCST the solution of Eq.(8) is simply $(\Delta p)^2 = m^2$, which is obviously unphysical, since the momentum Wilson loop is known to have only a finite number of discontinuities[6], while such a solution would imply that a photon is emitted or absorbed at any point of the loop. Therefore in this limit the transverse component of the loop area derivative always propagates.

Let us now further investigate Eq.(5). One can show that

$$\boldsymbol{\Delta} W\left[\mathbf{x}\right] = \int D\mathbf{p}\, \exp\left(i\int\limits_0^1 d\sigma' x_\mu \dot{p}_\mu\right) \int\limits_0^1 d\sigma \frac{\delta}{\delta p_\nu(\sigma)} \Delta p_\lambda \int\limits_{-0}^{+0} d\tau \tau \dot{p}_\nu \left(\sigma + \frac{1}{2}\tau\right) \dot{p}_\lambda\left(\sigma - \frac{1}{2}\tau\right) W\left[\mathbf{p}\right], \quad (9)$$

where $\boldsymbol{\Delta} = \int\limits_0^1 d\sigma \dot{x}_\nu \partial_\lambda^x \frac{\delta}{\delta\sigma_{\lambda\nu}(x)}$ is the functional Laplacian, and $\frac{\delta}{\delta p_\nu(\sigma)} \equiv \frac{1}{2}\left(\frac{\delta}{\delta p_\nu(\sigma+0)} + \frac{\delta}{\delta p_\nu(\sigma-0)}\right)$. Then, taking into account that $\dot{p}_\mu(\sigma) = \sum\limits_i \Delta p_\mu(\sigma_i)\delta\left(\sigma - \sigma_i\right)$, where σ_i's are the positions of the momentum discontinuities, and that when being applied to a functional without marked points $\boldsymbol{\Delta}$ may be rewritten as $\boldsymbol{\Delta} = \int\limits_0^1 d\sigma \int\limits_{\sigma-0}^{\sigma+0} d\sigma' \frac{\delta^2}{\delta x_\mu(\sigma)\delta x_\mu(\sigma')}$, we arrive by virtue of Eqs.(5) and (9) at the following equation for the momentum Wilson loop

$$\sum\limits_i \left(\Delta p\left(\sigma_i\right)\right)^2 W\left[\mathbf{p}\right] =$$

$$= 2e^2 \int\limits_0^1 d\sigma \frac{\delta}{\delta p_\nu(\sigma)} \Delta p_\lambda \left(\mathbf{1}_{\lambda\nu,\mu\rho} - \frac{(\Delta p)^2}{(\Delta p)^2 - \frac{Q[\mathbf{p}]}{e^2}\Lambda^2} \mathbf{P}_{\lambda\nu,\mu\rho}\right) \int D\mathbf{p}' T_{\mu\rho}\left[\mathbf{p} - \mathbf{p}'\right] W\left[\mathbf{p}'\right]. \quad (10)$$

Eq.(10) describes the random motion of the momentum loop in such a way that the free propagation of the loop is described by the factor $\sum\limits_i \left(\Delta p\left(\sigma_i\right)\right)^2$ (see Ref.6), while the R.H.S. of Eq.(10) describes the interaction of the momentum loop with the functional vorticity tensor current.

To conclude with, we shall investigate loop equation, which one can derive by substituting Eq.(5) into Eq.(9), in the coordinate representation. To this end we shall make use of the method of inversion of the functional Laplacian, which was developed in[8] and applied in[9] to the solution of the Cauchy problem for the loop equation in turbulence. This procedure yields the following equation for the Wilson loop in the coordinate representation

$$W\left[\mathbf{x}\right] = 1 + 2e^2 \int D\mathbf{p} \frac{1}{\int\limits_0^1 d\sigma' \int\limits_0^1 d\sigma'' \dot{p}_\alpha\left(\sigma'\right) G\left(\sigma' - \sigma''\right) \dot{p}_\alpha\left(\sigma''\right)} \cdot$$

$$\cdot \int\limits_0^1 d\sigma \Delta p_\lambda \left[2 \frac{\int\limits_0^1 d\sigma' \dot{G}\left(\sigma - \sigma'\right) \dot{p}_\nu\left(\sigma'\right) \left(\exp\left(i\int\limits_0^1 d\sigma'' x_\beta \dot{p}_\beta\right) - 1\right)}{\int\limits_0^1 d\sigma' \int\limits_0^1 d\sigma'' \dot{p}_\gamma\left(\sigma'\right) G\left(\sigma' - \sigma''\right) \dot{p}_\gamma\left(\sigma''\right)} - i\dot{x}_\nu(\sigma) \exp\left(i\int\limits_0^1 d\sigma' x_\beta \dot{p}_\beta\right)\right] \cdot$$

$$\cdot \left[\frac{(\Delta p)^2}{(\Delta p)^2 - \frac{Q[\mathbf{p}]}{e^2}\Lambda^2} \mathbf{P}_{\lambda\nu,\mu\rho} - \mathbf{1}_{\lambda\nu,\mu\rho}\right] \int D\mathbf{y}\, \exp\left(-i\int\limits_0^1 d\sigma' y_\zeta \dot{p}_\zeta\right) T_{\mu\rho}\left[\mathbf{y}\right] W\left[\mathbf{y}\right], \quad (11)$$

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where $G(\sigma - \sigma')$ is a certain smearing function. Therefore we have reduced Eq.(3) to a simple Volterra type-II linear integral equation, which can be treated perturbatively (notice, that in the low-energy limit the coupling constant in Eq.(11) stands only in front of the integral operator, since $\frac{\Lambda^2}{e^2}$ replaces by m^2, and the iterative procedure of the solution of Eq.(11) linearizes).

3. Conclusion

In this letter we have derived and investigated loop equation for the partition function of the 4D Euclidean UCST. Our main result is the reduction of this equation, given by formula (3), to a simple Volterra type-II integral equation, which is given by formula (11).

Besides that, we have investigated the obtained loop equation in the momentum representation. Namely, we have inverted the functional differential operator, standing on the L.H.S. of Eq.(3), and obtained the equation for the momentum area derivative of the Wilson loop, which is given by formula (5). This equation decouples into a pair of two independent equations (6) and (7) for the transverse and longitudinal parts of this area derivative, and one can see that at the value of the momentum discontinuity, satisfying Eq.(8), the transverse part does not propagate. In the low-energy limit of the UCST Eq.(8) has only a spurious solution, and thus in this limit the transverse component always propagates. The obtained equation for the momentum area derivative of the Wilson loop has been also applied to the derivation of Eq.(10), which describes the random motion of the momentum loop in such a way, that its L.H.S. describes the free propagation of the loop, while its R.H.S. describes the interaction of the loop with the functional vorticity tensor current.

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